Stockton University
Mathematical Mayhem 2018
Group Round - Solutions

April 14, 2018

Name: ______________________________

Name: ______________________________

Name: ______________________________

High School: ______________________________

Instructions:

- This round consists of **5** problems worth **16** points each for a total of **80** points.
- Each of the 5 problems is free response.
- Write your complete solution in the space provided including all supporting work.
- No calculators are permitted.
- This round is **75 minutes** long. **Good Luck!**

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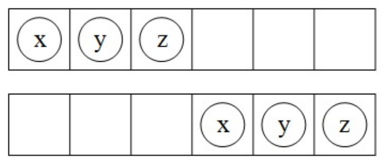
Problem #	1	2	3	4	5	Total
Points Earned						

♠ Group Round ♠

Problem 1. To start, three coins are placed in the first three of six squares as shown in the upper diagram. A move consists of moving one coin one space to the right. A move can only be performed if the space to the right of the coin that is moving is not occupied by another coin. How many different sequences of moves can be used to move the three coins from the first three squares to the last three squares, from the configuration in the upper diagram to the configuration in the lower diagram?



Solution to Question 1. Each set of moves can be represented by a length 9 string of X's, Y's, and Z's in which the letter indicates the coin moved. The first letter must be Z. We approach this by a brute force case by case analysis, first separating cases by the length of the initial substring of Z's.

3 Z's If we start with 3 Z's, all that remains is a length 6 string of X's and Y's in which no initial substring can have more X's than Y's. Again, we separate into cases based on the length of the initial substring of Y's. There is only one length 6 string starting with three Y's, YYYXXX. There are two strings that start with two Y's: YYXYXX and YYXXYX. There are two strings that start with one Y: YXYYXX and YXYXYX. Hence, there are 5 length nine strings beginning with ZZZ.

2 Z's If we start with two Z's, we are left with a length 7 string with one Z, three Y's, and 3 X's. These are precisely the strings we counted in the previously case with a Z inserted. The final Z cannot be the first letter of the string, as it would duplicate the ZZZ case, and it must appear before the final Y. In the table below, we mark the allowed positions for the last Z with $*$'s.

String	Allowed Z Positions
Y $*$ Y $*$ YXXX	2
Y $*$ Y $*$ X $*$ YXX	3
Y $*$ Y $*$ X $*$ X $*$ YX	4
Y $*$ X $*$ Y $*$ YXX	3
Y $*$ X $*$ Y $*$ X $*$ YX	4

This yields 2 + 3 + 4 + 3 + 4 = 16 strings in this case.

1 Z Following a similar idea to the 2Z case, we note that we must examine how to insert pairs of Z's into those same length six lists of Y's and X's so that the first Z appears before the second Y and the last Z appears before the last Y. First, we place the first of the two Z's. This Z must appear before the

second Y but not in the first positione.

String	Allowed First Z Positions
$\mathrm{Y} * \mathrm{YYXXX}$	1
$\mathrm{Y} * \mathrm{YXYXX}$	1
$\mathrm{Y} * \mathrm{YXXYX}$	1
$\mathrm{Y} * \mathrm{X} * \mathrm{YYXX}$	2
$\mathrm{Y} * \mathrm{X} * \mathrm{YXYX}$	2

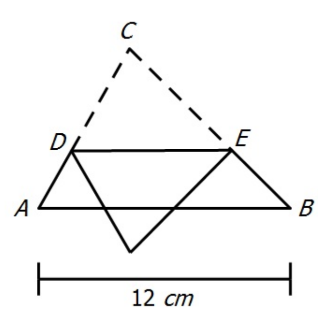
There are $1 + 1 + 1 + 2 + 2 = 7$ positions for the first Z. We now place the second Z, and consider the seven length 7 strings made by placing the first Z as shown in the previous table. This time the * will represent possible positions of the second Z, which must be after the first Z but before the last Y.

String	Allowed First Z Positions
$\mathrm{YZ} * \mathrm{Y} * \mathrm{YXXX}$	2
$\mathrm{YZ} * \mathrm{Y} * \mathrm{X} * \mathrm{YXX}$	3
$\mathrm{YZ} * \mathrm{Y} * \mathrm{X} * \mathrm{X} * \mathrm{YX}$	4
$\mathrm{YZ} * \mathrm{X} * \mathrm{Y} * \mathrm{YXX}$	3
$\mathrm{YXZ} * \mathrm{Y} * \mathrm{YXX}$	2
$\mathrm{YZ} * \mathrm{X} * \mathrm{Y} * \mathrm{X} * \mathrm{YX}$	4
$\mathrm{YXZ} * \mathrm{Y} * \mathrm{X} * \mathrm{YX}$	3

Note that there are $2 + 3 + 4 + 3 + 2 + 4 + 3 = 21$ length 9 strings that begin with only one Z.

In total we have $5 + 16 + 21 = 42$ ways to move the pennies.

Problem 2. The base of a triangular piece of paper ABC is 12 cm long. The paper is folded down over the base, with the crease DE parallel to the base of the paper. The area of the triangle that is below the base after this fold occurs is 16% of the area of the triangle ABC. What is the length of DE? See the figure below which may not be drawn to scale.



Solution to Question 2. Let Z be the bottommost corner of the triangle that is below the fold. Let X and Y be the points at which line segment AB intersects with line segments DZ and EZ respectively. Then the area of $\triangle XYZ$ is 16% of the area of $\triangle ABC$. Notice folding in DE constitutes reflection in DE, and that reflection preserves angle measure. So, $\triangle ACB$ is similar to $\triangle XZY$ since $\angle XZY$ is $\angle ACB$ after reflection and since

$$\angle XYZ = \angle EYB = \angle DEY = \angle CED = \angle CBA$$

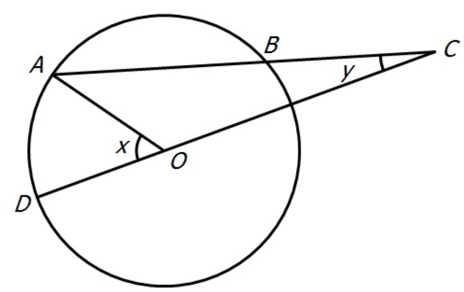
by the Alternate Interior Angle Theorem and reflection. Since $\triangle XZY$ is similar to $\triangle ACB$ and its area is $.16 = (.4)^2$ that of $\triangle ACB$, the sides of $\triangle XZY$ are .4 times as long as the sides of $\triangle ACB$.

Draw CZ and let P be the intersection of CZ with $\overleftrightarrow{AB}$ while Q is the intersection of of CZ with $\overleftrightarrow{DE}$. Note that CP is an altitude of $\triangle ACB$. Now,

$$CP = CQ + QP = ZQ + QP = ZP + 2PQ.$$

Since the sides of $\triangle XZY$ are .4 times as long as the sides of $\triangle ACB$, then $ZP = 0.4CP$. Since $CP = ZP + 2PQ$, then $PQ = 0.3CP$, and so $CQ = CP - PQ = 0.7CP$. Since CQ is 0.7 times the length of CP, the line segment DE is 0.7 times the length of AB, again by similar triangles, so $DE = 0.7(12) = 8.4$.

Problem 3. In a circle with center O chord AB is produced so that the length of BC equals the radius of the circle. The segment CO extends to D, a point on the circle. What is the relationship between angles x and y? That is, write an equation for x in terms of y or vice versa.



Solution to Question 3. Connect the radius from O to B. Then $\triangle BOC$ is an isosceles triangle. $\angle OAB = \angle OBA = 2y$. $x = \angle OAB + y = 3y$.

Problem 4. If $n = abc$ is a positive three-digit number with digits a, b, and c (not to be confused with the product $a \cdot b \cdot c$), let

$$f(n) = a + b + c + a \cdot b + a \cdot c + b \cdot c + a \cdot b \cdot c.$$

For example, $f(234) = 2 + 3 + 4 + 6 + 8 + 12 + 24$. How many different three digit positive integers satisfy the equation $f(n) = n$ and what are those integers?

Solution to Question 4. Note that $f(n) = (a+1)(b+1)(c+1) - 1$ and so we must have $(a+1)(b+1)(c+1) - 1 = 100a + 10b + c$. We can rewrite this as $a(b+1)(c+1) + b(c+1) = 100a + 10b$. Since $b, c \leq 9$ we have $(b+1)(c+1) \leq 100$ and $c+1 \leq 10$. Equality is obtained only when $b = c = 9$. However, a can be any digit 1 to 9, and hence there are 9 solutions: 999, 899, 799, 699, 599, 499, 399, 299, and 199.

Problem 5. You walk up a flight of 20 stairs, going up either 1 or 2 stairs with each step. There is a snake on the 9th stair and a snake on the 16th stair, so you cannot step on either of those stairs. In how many different ways can you walk up the stairs? For example, one way is 2-2-2-2-2-1-2-2-2-2-1 and another is 1-1-1-1-1-1-1-1-2-1-1-1-1-1-2-1-1-1.

Solution to Question 5. Let S_n be the number of ways to walk up a set of n unobstructed stairs. Note that

$$S_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}.$$

Further, we must step from step 8 to 10 and from step 15 to 17. Our options are the product of the number of ways to climb stairs 0 to 8, the numbers of ways to climb stairs 10 to 15, and the number of ways to climb stairs 17 to 20. That is, the solution is

$$\begin{aligned} S_8 S_5 S_3 &= \left[\binom{8}{0} + \binom{7}{1} + \binom{6}{2} + \binom{5}{3} + \binom{4}{4}\right]\left[\binom{5}{0} + \binom{4}{1} + \binom{3}{2}\right]\left[\binom{3}{0} + \binom{2}{1}\right] \\ &= (1 + 7 + 15 + 10 + 1)(1 + 4 + 3)(1 + 2) \\ &= 34 \cdot 8 \cdot 3 = 816 \end{aligned}$$