Stockton University
Mathematical Mayhem 2019
Individual Round - Solutions

March 30, 2019

Instructions:

- This round consists of **18** problems worth a total of **80** points, made up of 8 Appetizers worth 3 points each, 7 Entrées worth 5 points each, and 3 Desserts worth 7 points each.
- Each of the 18 problems is multiple choice, and each problem comes with **5** possible answers.
- For each problem, **mark your answer on the answer sheet**.
- You are not required to show any work this round.
- No calculators are permitted.
- This round is **75 minutes** long. **Good Luck!**

♣ Appetizers ♣

Problem 1. Suppose a, b, and c are positive integers with $a < b < c$ such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

What is $a + b + c$?
(A.) no such integers a, b, and c exist (B.) $\boxed{11}$ (C.) 9 (D.) 4 (E.) 1

Solution to Question 1. This is can be done by inspection. More systematically, note that we must have $1/a < 1$, so $a > 1$. Since $1/a > 1/b > 1/c$, we must also have $1/a > 1/3$; so $a < 3$. Thus, $a = 2$. Now $1/b + 1/c = 1/2$ where $2 < b < c$. Similar to before, $1/b > 1/4$, so $b < 4$. Thus, $b = 3$. With $a = 2$ and $b = 3$ we have $1/2 + 1/3 + 1/c = 1$ which is satisfied when $c = 6$ and $a + b + c = 2 + 3 + 6 = 11$.

Problem 2. Tom is trying to sleep. He counts one sheep every 3 seconds, and he counted his first sheep of the night 15 seconds after 10:20 pm. He just counted his 235th sheep. What time is it to the nearest minute?
(A.) 10:24 pm (B.) 10:27 pm (C.) 10:30 pm (D.) $\boxed{10:32\,\text{pm}}$ (E.) 10:34 pm

Solution to Question 2. Tom counted his n-th sheep $3n$ seconds after 10:20 and 12 seconds. That occurs 705 seconds after 10:20 and 12 seconds. That is 11 minutes and 45 seconds after 10:20 and 12 seconds, so at 10:31 and 57 seconds. It is closest to 10:32 pm.

Problem 3. The symbols $\heartsuit$ and $\spadesuit$ represent two different positive integers less than 20. If $\heartsuit \times \heartsuit \times \heartsuit = \spadesuit$, what is the value of $\spadesuit \times \spadesuit$?
(A.) 12 (B.) 16 (C.) 36 (D.) $\boxed{64}$ (E.) 81

Solution to Question 3. The only cubes of positive integers that are less than 20 are 1 and 8. Since $1^3 = 1$ and $2^3 = 8$. The first equation leads to $\heartsuit = 1 = \spadesuit$. Since $\heartsuit$ and $\spadesuit$ must represent two different positive integers we must use the second equation and so $\heartsuit = 2$ and $\spadesuit = 8$. Hence, $\spadesuit \times \spadesuit = 64$.

Problem 4. A circle of area 48π is inscribed in an equilateral triangle. What is the perimeter of this triangle?
(A.) $72\sqrt{3}$ (B.) $48\sqrt{3}$ (C.) 36 (D.) 24 (E.) $\boxed{72}$

Solution to Question 4. Note if a is the given area and r is the radius of the inscribed circle, then $a = \pi r^2$ and so $r = 4\sqrt{3}$. The medians of this triangle intersect at the center of the inscribed circle. Since the radius of the circle is the distance from the center of the circle to a side of the triangle and the height h of the triangle is one of the medians, we have $r = \frac{h}{3}$ and $h = 12\sqrt{3}$. We can solve for the length of a side s via the Pythagorean Theorem, noting that $h^2 + (s/2)^2 = s^2$, we have that $h^2 = (3/4)s^2$ and so $(2\sqrt{3}/3)h = s = 24$. Lastly, the perimeter is $3s = 72$.

Problem 5. In the equation $2x^2 - hx + 2k = 0$, the sum of the roots is 4 and the product of the roots is -3. What are the values of h and k respectively?
(A.) 8 and -6 (B.) 4 and -3 (C.) -3 and 4 (D.) -3 and 8 (E.) $\boxed{8 \text{ and } -3}$

Solution to Question 5. Let a and b be the roots, so $a + b = 4$ and $ab = -3$. Further,

$$2x^2 - hx + 2k = 2(x^2 + (-h/2)x + k) = 2(x - a)(x - b) = 2(x^2 + (-a - b)x + ab) = 0.$$

That is, $-a - b = -4 = -h/2$ giving $h = 8$ and $ab = k = -3$.

Problem 6. A large cylindrical vase with diameter 6 inches and height 16 inches is filled with water up to an inch from its top. The water in the vase is used to fill smaller vases that are 3 inches in diameter and 6 inches tall up to an inch from their tops. How many smaller vases can it fill?
(A.) $\boxed{12}$ (B.) 135π (C.) 6 (D.) 10 (E.) 8

Solution to Question 6. The volume of the water in the large vase is $\pi(6/2)^2(15)\ in^3$ and each small vase contains $\pi(3/2)^2(5)\ in^3$. Note that

$$\frac{\pi(6/2)^2(15)}{\pi(3/2)^2(5)} = 12.$$

Problem 7. What is the angle formed by the hands of a clock at 2:15?
(A.) $172\frac{1}{2}^\circ$ (B.) $157\frac{1}{2}^\circ$ (C.) 30° (D.) $27\frac{1}{2}^\circ$ (E.) $\boxed{22\frac{1}{2}^\circ}$

Solution to Question 7. Notice that the minute hand points to the 3 and the hour hand is $1/4$ of the way between the 2 and 3. The angle between the 2 and 3 is $360^\circ/12 = 30^\circ$ and the so the angle between the hands is $(3/4)30^\circ = 22\frac{1}{2}^\circ$.

Problem 8. Let $f(x)$ be a function defined on the integers where

$$f(x) = \begin{cases} \frac{x-1}{2} & \text{if x is odd, or} \\ x^2 + 1 & \text{if x is even.} \end{cases}$$

What is $f(f(f(f(f(5)))))$?
(A.) 0 (B.) 1 (C.) $\boxed{2}$ (D.) 4 (E.) 5

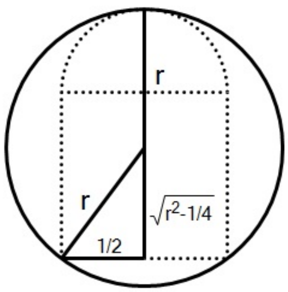
Solution to Question 8. Note that $f(5) = 2$ and $f(2) = 5$. So $f(f(f(f(f(5))))) = 2$.

♢ Entrées ♢

Problem 9. A silo-shaped figure is formed by placing a semicircle of diameter 1 on top of a unit square, with the diameter coinciding with the top of the square. What is the radius of the smallest circle which contains this figure?

(A.) $1/2$ (B.) $2/3$ (C.) $\boxed{5/6}$ (D.) 1 (E.) $7/6$

Solution to Question 9. The radius of the large circle r satisfies $r + \sqrt{r^2 - 1/4} = 3/2$, as shown in the diagram. So $r^2 - 1/4 = (3/2 - r)^2$, and so $3r = 10/4$ or $r = 10/12$.



Problem 10. Patty randomly selects an integer from 1 to 9. Mark, independent of Patty's choice, also randomly selects an integer from 1 to 9. What is the most likely value for the units digit of the sum of Mark and Patty's numbers?

(A.) $\boxed{0}$ (B.) 1 (C.) 9 (D.) 2 (E.) 8

Solution to Question 10. Out of the 81 equally likely pairs, the following displays the number of ways each sum can occur:

Sum	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Occurrence	1	2	3	4	5	6	7	8	9	8	7	6	5	4	3	2	1

The last digit 0 occurs in nine ways, while any other last digit occurs in eight ways.

Problem 11. If $\sqrt{x-1} - \sqrt{x+1} + 1 = 0$, then to which value is $4x$ equal?

(A.) $4\sqrt{-1}$ (B.) $\boxed{5}$ (C.) 0 (D.) $1\frac{1}{4}$ (E.) no real value

Solution to Question 11. Note that

$$\begin{aligned} \sqrt{x-1} + 1 &= \sqrt{x+1} \\ x - 1 + 2\sqrt{x-1} + 1 &= x + 1 \\ 2\sqrt{x-1} &= 1 \\ 4x - 4 &= 1, \text{ and lastly} \\ 4x &= 5. \end{aligned}$$

Problem 12. How many ways are there to walk up a set of 10 stairs if in each step you take you can step up one stair or two stairs?

(A.) 55 (B.) 85 (C.) $\boxed{89}$ (D.) 100 (E.) 144

Solution to Question 12. Let a_n be the number of ways that there are to walk up n stairs. Note that for all $n \geq 3$, we have $a_n = a_{n-1} + a_{n-2}$; in order to walk up n stairs, you can walk up $n-1$ stairs and then take a step up one stair or you can walk up $n-2$ stairs and then take a step up two stairs. Further, $a_1 = 1$ and $a_2 = 2$, and so this gives the Fibonacci sequence:

$$\begin{array}{ccccccccc} a_3 & = & 3 & a_4 & = & 5 & a_5 & = & 8 \\ a_6 & = & 13 & a_7 & = & 21 & a_8 & = & 34 \\ a_9 & = & 55 & a_{10} & = & 89 & & & \end{array}$$

Problem 13. Two men set out at the same time to walk towards each other from 72 miles apart. The first man walks at the rate of 4 mph. The second man walks 2 miles the first hour, 2.5 miles the second hour, 3 miles the third hour and so on each hour walking .5 miles more than he did in the previous hour. What is the distance that the second man has walked when the two men meet?

(A.) 44 miles (B.) 40 miles (C.) $\boxed{\text{36 miles}}$ (D.) 32 miles (E.) 28 miles

Solution to Question 13. Let t be the number of hours each man walked. Then the distance the second man walked is given by

$$S = \sum_{i=1}^{t} \left(2 + \frac{i-1}{2}\right) = \sum_{i=1}^{t} \left(\frac{3}{2} + \frac{i}{2}\right) = \frac{3t}{2} + \frac{(t)(t+1)}{4} = \frac{7t + t^2}{4}.$$

Since the first man walked $4t$ miles, we have $\dfrac{7t + t^2}{4} + 4t = 72$. Solving for t gives $t = 9$ and so each man has walked 36 miles.

Problem 14. Suppose that x is a two digit number and that y is the number formed by reversing the digits of x. The number $x^2 - y^2$ must be divisible by each of the following numbers *except*

(A.) $\boxed{\text{the product of the digits of x}}$ (B.) 9 (C.) the sum of the digits of x

(D.) the difference of the digits of x (E.) 11

Solution to Question 14. Let t and u be the digits of x so that $x = 10t + u$ and $y = 10u + t$. Then

$$(10t + u)^2 - (10u + t)^2 = 99(t^2 - u^2) = 9 \cdot 11(t - u)(t + u),$$

which is divisible by $9, 11, t + u$, and $t - u$ but not necessarily tu. As a counter counterexample, note $51^2 - 15^2 = 2601 - 225 = 2376$ is not divisible by $5 = 5 \cdot 1$.

Problem 15. If n is a positive integer, the symbol $n!$ represents the product of the integers from 1 to n. For example, $4! = (1)(2)(3)(4)$ or $4! = 24$. If x and y are integers and

$$\frac{30!}{36^x\, 25^y}$$

is equal to an integer, what is the maximum possible value of $x + y$?

(A.) $\boxed{10}$ (B.) 47 (C.) 17 (D.) 26 (E.) 13

Solution to Question 15. Note that $30! = 2^{26}3^{14}5^7 7^4 11^2 13^2 17^1 19^1 23^1 29^1$. So,

$$\begin{aligned}\frac{30!}{36^x\, 25^y} &= \frac{30!}{(2^2\, 3^2)^x\, (5^2)^y} \\ &= \frac{2^{26}3^{14}5^7 7^4 11^2 13^2\, 17^1\, 19^1\, 23^1\, 29^1}{2^{2x}\, 3^{2x}\, 5^{2y}} \\ &= 2^{26-2x}3^{14-2x}5^{7-2y}7^4 11^2 13^2 17^1 19^1 23^1 29^1\end{aligned}$$

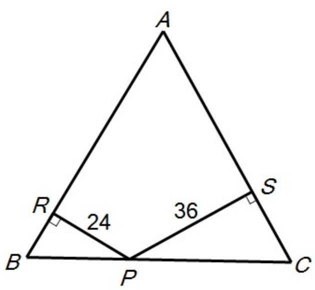
In order for $\frac{30!}{36^x\, 25^y}$ to be an integer, we must have $26 - 2x \geq 0$, $14 - 2x \geq 0$ and $7 - 2y \geq 0$. The largest possible integer values for x and y that satisfy these inequalities are 7 and 3 respectively. So $x + y = 10$.

♡ Desserts ♡

Problem 16. Consider a herd of rhinoceros that stretches for three miles and travels at a constant rate across the African savannah. An oxpecker bird flies at a constant (though different) rate from the rear of the herd to the front of the herd and back again. If the herd travels 6 miles in the time that it takes the bird to fly forward and back, what is the total distance the bird travels in miles?
(A.) $5\sqrt{3}$ (B.) $12 - 3\sqrt{3/4}$ (C.) $\boxed{3 + 3\sqrt{5}}$ (D.) $6\sqrt{3}$ (E.) $6 + 3\sqrt{3}$

Solution to Question 16. Let q be the distance traveled by the bird to get to the front of the herd and let p be the distance traveled by the herd during the same time. We have $q = p + 3$. To reach the rear of the herd when the rear has advanced 6 miles from the starting point, the bird needs to travel $q - 6$ miles back (after reaching the front). As a fraction of the forward journey, we have $(q - 6)/q$. Thus the herd will move $p(q - 6)/q$ while the bird is returning to the rear. We have $6 = p + p(q - 6)/q = q - 3 + (q - 3)(q - 6)/q$. So $q = 9 + 3\sqrt{5}$ and hence $p = 3 + 3\sqrt{5}$.

Problem 17. Triangle ABC is isosceles with $AB = AC$ and $BC = 65$. P is a point on BC such that the perpendicular distances from P to AB and AC are 24 and 36 respectively (these are RP and SP in the figure). What is the area of triangle $\triangle ABC$?
(A.) 1254 (B.) 1640 (C.) 1950 (D.) $\boxed{2535}$ (E.) 2942



Solution to Question 17. Note that since $AB = AC$, we have $\angle ABC = \angle ACB$. Also, $\angle ARP = \angle CSP$ and hence $\triangle BRP \sim \triangle CSP$. Hence $PC/PB = PS/PR = 36/24 = 1.5$ and $PB + PC = 65$. Solving these equations, we see that $2.5PB = 65$ and so $PB = 26$ and $PC = 39$. Applying the Pythagorean Theorem, we can see $PB^2 = RB^2 + RP^2$ and so $26^2 = RB^2 + 24^2$. That is, $RB^2 = 100$ and $RB = 10$ and further since $SC/RB = PS/PR = 1.5$ we have $SC = 15$.

Continuing, $AR^2 + 24^4 = AS^2 + 36^2$ and $AR + RP = AS + SC$. That is, $AR - AS = 5$ and $AR^2 - AS^2 = 36^2 - 24^2 = (60)(12) = 720$. Putting this together, $AR^2 - AS^2 = 720 = (AR - AS)(AR + AS) = 5(AR + AS)$ and $AR + AS = 144$. Hence, $2AR = 149$ and $AR = 74.5$. That is, $AB = 84.5 = AC$.

Now, the area of $\triangle ABC$ can be found by adding the areas of $\triangle ABP$ and $\triangle APC$. Those are $(24)(84.5)/2$ and $(36)(84.5)/2$ respectively. Hence,

$$\text{Area}\,\triangle ABC = (24)(84.5)/2 + (36)(84.5)/2 = 30(84.5) = 2535.$$

Problem 18. For any real number x the floor of x, denoted by $\lfloor x \rfloor$, is the largest integer less than or equal to x. For example, $\lfloor 4.2 \rfloor = 4$, $\lfloor .9 \rfloor = 0$, $\lfloor -3.45 \rfloor = -4$, and $\lfloor 2 \rfloor = 2$. The number S is the sum of all integers k where k is divisible by $\lfloor \sqrt{k} \rfloor$ and $1 \le k \le 999999$. What is the value of S?

(A.) 999500000 (B.) $\boxed{999999000}$ (C.) 998999500 (D.) 999000000 (E.) 998500500

Notice that

$$\begin{array}{lclclclclcl}
\lfloor\sqrt{1}\rfloor & = & 1 & \quad & \lfloor\sqrt{2}\rfloor & = & 1 & \quad & \lfloor\sqrt{3}\rfloor & = & 1 \\
\lfloor\sqrt{4}\rfloor & = & 2 & & \lfloor\sqrt{5}\rfloor & = & 2 & & \lfloor\sqrt{6}\rfloor & = & 2 \\
\lfloor\sqrt{7}\rfloor & = & 2 & & \lfloor\sqrt{8}\rfloor & = & 2 & & \lfloor\sqrt{9}\rfloor & = & 3 \\
\lfloor\sqrt{10}\rfloor & = & 3 & & \lfloor\sqrt{11}\rfloor & = & 3 & & \lfloor\sqrt{12}\rfloor & = & 3 \\
\lfloor\sqrt{13}\rfloor & = & 3 & & \lfloor\sqrt{14}\rfloor & = & 3 & & \lfloor\sqrt{15}\rfloor & = & 3 \\
\lfloor\sqrt{16}\rfloor & = & 4 & & \lfloor\sqrt{17}\rfloor & = & 4 & & \lfloor\sqrt{18}\rfloor & = & 4
\end{array}$$

That is to say, the positive integers n so that $\lfloor \sqrt{n} \rfloor = k$ are those so that $k^2 \le n < (k+1)^2$. Since k^2 is always divisible by k, the values of k where k is divisible by $\lfloor \sqrt{k} \rfloor$ are the multiples of k starting with k^2 and ending with $k^2 + 2k < k^2 + 2k + 1 = (k+1)^2$. To see this, note that the first few terms in the sum S are

$$S = 1 + 2 + 3 + 4 + 6 + 8 + 9 + 12 + 15 + 16 + 20 + 24 + 25 + 30 + 35 + \dots.$$

That is,

$$\begin{aligned}
S &= \sum_{i=1}^{\lfloor\sqrt{999999}\rfloor} \left(i^2 + i^2 + i + i^2 + 2i\right) \\
&= \sum_{i=1}^{999} \left(3i^2 + 3i\right) \\
&= 3\left(\sum_{i=1}^{999} i^2 + \sum_{i=1}^{999} i\right) \\
&= 3\left(\frac{(2(999)+1)(999+1)(999)}{6} + \frac{(999+1)(999)}{2}\right) \\
&= \frac{(1999)(1000)(999)}{2} + \frac{(3)(1000)(999)}{2} \\
&= 500((1999)(999) + (3)(999)) \\
&= 500(999)(2002) \\
&= 1000(999)(1001) \\
&= 1000(999999) \\
&= 999999000
\end{aligned}$$