The Richard Stockton College of New Jersey
Mathematical Mayhem 2014

Group Round - Solutions

March 22, 2014

Name: ____

Name: ____

Name: ____

High School: ____

Instructions:

- This round consists of **5** problems worth **16** points each for a total of **80** points.
- Each of the 5 problems is free response.
- Write your complete solution in the space provided including all supporting work.
- No calculators are permitted.
- This round is **75 minutes** long. **Good Luck!**

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Problem #	1	2	3	4	5	Total
Points Earned						

♠ Group Round ♠

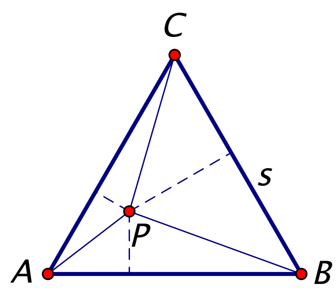
Problem 1. A point is selected at random inside an equilateral triangle. From this point perpendiculars are dropped to each side. Show the sum of theses perpendiculars is equal to the altitude of the triangle.

Solution to Question 1. Let P be any point inside the equilateral $\triangle ABC$ with sides length s and let the three perpendicular be P_a, P_b, P_c.

$$\text{Area of } \triangle ABC = \text{ Area of } \triangle APB + \text{ Area of } \triangle BPC + \text{ Area of } \triangle CPA$$

$$= \frac{1}{2}s(P_a + P_b + P_c).$$

On the other hand, the area of $\triangle ABC = \frac{1}{2}sh$, where h is the altitude of $\triangle ABC$. Hence $h = P_a+P_b+P_c$.



Problem 2. Tom, Dick, and Harry began a 100-mile journey at the same time. Tom and Harry went by car at the speed of 25 mph, while Dick walked at the speed of 5 mph. After a certain distance, Harry got out of the car and walked on at 5 mph, while Tom went back to pick up Dick and got him to the destination at the same time that Harry arrived. Find the number of hours required for the entire journey.

Solution to Question 2. Let t_1, t_2, t_3 be the number of hours, respectively, that the car travels forward, back to pick up Dick, then forward to the destination. Then we may write

$$25t_1 - 25t_2 + 25t_3 = 100 \text{ for the car,}$$

$$5t_1 + 5t_2 + 25t_3 = 100 \text{ for Dick,}$$

$$25t_1 + 5t_2 + 5t_3 = 100 \text{ for Harry,}$$

which can be reduced to the system of

$$t_1 - t_2 + t_3 = 4,$$

$$t_1 + t_2 + 5t_3 = 20,$$

$$5t_1 + t_2 + t_3 = 20,$$

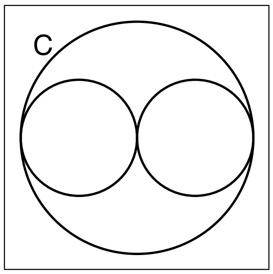
with solution $t_1 = 3, t_2 = 2, t_3 = 3$. Therefore, $t_1 + t_2 + t_3 = 8$ hours for the entire journey.

Problem 3. Start with a circle C with two smaller circles inscribed in it as the initial configuration, as shown in Figure (a) below. Consider the procedure *Add Circles* defined as follows.

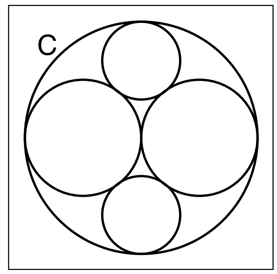
Add Circles: Inscribe a circle into each non circular region.

The result of applying *Add Circles* to Figure (a) is Figure (b), shown below. Now repeat *Add Circles* four more times; in total, you have applied *Add Circles* to the initial circle configuration five times. After the fifth application of *Add Circles*, how many of the total circles pictured do not touch circle C?

Suppose *Add Circles* was applied to the initial circle configuration a total of n times. After the n-th application of *Add Circles*, how many of the total circles pictured do not touch circle C?



(a) Initial Circle Configuration



(b) Result of applying *Add Circles* to Initial Circle Configuration

Solution to Question 3.

Each step triples the numbers of non-circular regions. Since 2 circles are added in the first application of *Add Circles*, the total number of circles added in the first 5 applications of *Add Circles* is

$$2 + 6 + 18 + 54 + 162 = 242.$$

The number of added circles which touch C doubles at each step. So, since 2 circles that touch C were added in the first application of *Add Circles*, the total number of circles added in the first 5 applications of *Add Circles* that touch C is

$$2 + 4 + 8 + 16 + 32 = 62.$$

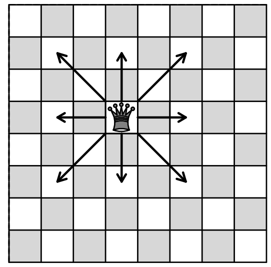
Thus, the difference $242 - 62 = 180$ is the number of circles added in the first 5 applications of *Add Circles* that don't touch C. Notice the original two circles are irrelevant to this count. Generalizing, the total number of circles added in the first n applications of *Add Circles* that don't touch C is

$$2(1 + 3 + 3^2 + ... + 3^{n-1}) - 2(1 + 2 + 2^2 + ... + 2^{n-1}).$$

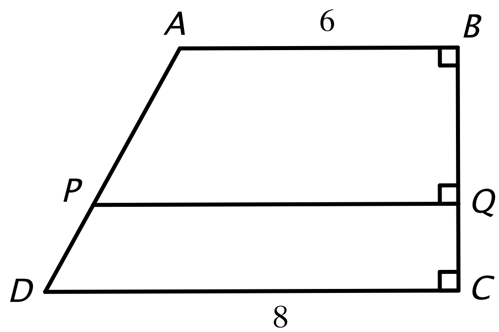
Problem 4. The game of chess is played on an 8×8 board. The queen is a piece in chess that can move any number of squares in any of the eight directions shown in the figure below.

What is the maximum number of queens that can be placed on a chess board so that none of the queens can capture another in a single move? Explain why this is the maximum, and find (and draw) a configuration of that number of queens so that none of the queens can capture another in a single move.

Solution to Question 4. Up to rotation and reflection, there are twelve unique placements of 8 queens on a chess board that meet the criteria. The proof that you cannot fit a 9th queen is an application of the pigeonhole principle, as two queens must share a column (or row).



Problem 5. *ABCD* is a trapezoid with parallel sides *AB* and *DC* both of which are perpendicular to side *BC*. The line *PQ* is parallel to *AB* and divides the trapezoid into two regions of equal area. If *AB* = 6, *DC* = 8, how long is segment *PQ*?



Solution to Question 5. Let the length of *PQ* be x and the length of *BC* be ℓ, which we tacitly assume is greater than 0. Then the length of *BQ* is $\ell\left(\frac{x-6}{2}\right)$ while the length of *QC* is $\ell\left(\frac{8-x}{2}\right)$. This gives that the area of trapezoid *ABQP* is $\ell\left(\frac{x-6}{2}\right)\left(\frac{x+6}{2}\right)$ while the area of trapezoid *DCQP* is $\ell\left(\frac{8-x}{2}\right)\left(\frac{8+x}{2}\right)$. Hence,

$$\begin{aligned} \ell\left(\frac{x-6}{2}\right)\left(\frac{x+6}{2}\right) &= \ell\left(\frac{8-x}{2}\right)\left(\frac{8+x}{2}\right) \\ (x-6)(x+6) &= (8-x)(8+x) \\ x^2-36 &= 64-x^2 \\ x &= \pm 5\sqrt{2}. \end{aligned}$$

Given our context, the length of PQ is $\boxed{x = 5\sqrt{2}}$.